Transformations of real-time finite-temperature Feynman rules

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Abstract

We consider transformations of the 2×2 propagator matrix in real-time finite-temperature field theory, resulting in transformed n–point functions. As special cases of such a transformation we examine the Keldysh basis, the retarded/advanced RA basis, and a Feynman-like $F\bar{F}$ basis, which differ in this context as to how "economically" certain constraints on the original propagator matrix elements are

implemented. We also obtain the relation between some of these real-time functions and certain analytic continuations of the imaginary-time functions. Finally, we compare some aspects of these bases which arise in practical calculations.

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1 Introduction

The fields $\phi(x) = \phi(t, \vec{x})$ describing particles in a heat bath at temperature β^{-1} are either periodic (bosons) or anti-periodic (fermions) in the complex time plane, with period $-i\beta$ [1–5]. Integrals over time then follow a path C in the complex time plane connecting a point $-T$ to a point $-T - i\beta$. In the imaginary-time formalism the straight path connecting these points is chosen, resulting in time arguments $t = -i\tau$ that are purely imaginary – an analytic continuation is then involved for quantities defined for real times.

In real-time formalisms a path which includes the real time axis is chosen, and thermal Green functions are then time-ordered with respect to this path:

$$G_C(x_1, \ldots, x_n) = \langle T_C \phi(x_1) \cdots \phi(x_n) \rangle \,. \tag{1}$$

A standard choice of this path is the concatenation of the following four pieces: $C_I : \; -T \to T$; $\; C_{III} : \; T \to T - i\sigma$; $\; C_{II} : \; T - i\sigma \to -T - i\sigma$; $\; C_{IV} : \; -T - i\sigma \to -T - i\beta$, where σ is an arbitrary parameter $0 \leq \sigma \leq \beta$ [2, 4]. As $T \to \infty$ the contributions from C_{III} and C_{IV} can be neglected in the majority of cases. It is then convenient to introduce the thermal doublet

$$\phi_a = \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix}, \tag{2}$$

where $a, b = +, -$. The fields $\phi_+(x) = \phi(x)$ and $\phi_-(x) = \phi(t - i\sigma, \vec{x})$ are defined on the forward and backward segments of the time contour, C_I and C_{II} respectively. This leads to a matrix form of the thermal Green function:

$$G_{a_1 \cdots a_n}(x_1, \ldots, x_n) = \langle T_C \phi_{a_1}(x_1) \cdots \phi_{a_n}(x_n) \rangle \,. \tag{3}$$

The dependence on σ will not be indicated explicitly, except in cases where a particular value applies. Two popular choices for this contour parameter that have been studied are $\sigma = 0$, which is the closed time-path formalism of Schwinger, Keldysh and others [6–9], and $\sigma = \frac{1}{2}$, which is equivalent to the "unitary" formulation of thermo field dynamics [4].

The propagator for free particles is a 2×2 matrix and in momentum space it has the form

$$D_{ab}(p) = \begin{pmatrix} D_{++}(p) & D_{+-}(p) \\ D_{-+}(p) & D_{--}(p) \end{pmatrix}, \tag{4}$$

where

$$
\begin{aligned}
D_{++}(p) &= D^*_{--}(p) = \\
&= \theta(p_0)\Delta_F(p) - \theta(-p_0)\Delta^*_F(p) + \eta\varepsilon(p_0)n(x)[\Delta_F(p) + \Delta^*_F(p)], \\
D_{+-}(p) &= \eta e^{2\sigma p_0 - x} D_{-+}(p) = \\
&= \eta\varepsilon(p_0)n(x)e^{\sigma p_0}[\Delta_F(p) + \Delta^*_F(p)].
\end{aligned} \tag{5}
$$

Here $\Delta_F(p)$ is the Feynman propagator ($i/[p^2 - m^2 + i\varepsilon]$ for scalar bosons), and

$$
n(x) = \frac{1}{e^x - \eta} \tag{6}
$$

is the Bose-Einstein (Fermi-Dirac) distribution function with the properties

$$
\begin{aligned}
e^x n(x) &= -\eta n(-x), \\
n(x) + n(-x) &= -\eta,
\end{aligned} \tag{7}
$$

where $x = \beta(p_0 - \mu)$, μ is the chemical potential, and $\eta = +1\,(-1)$ for bosons (fermions).

In many applications Green functions other than these time-ordered products arise. A common example is the linear response of the field to a weak external current [1, 9], which involves the retarded propagator Δ_R. This product also arises in a comparison of certain analytic continuations of the n–point functions in the imaginary-time formalism to functions in real-time formalisms [2, 10, 11, 12, 13]. As the retarded propagator can be written as

$$
\Delta_R(p) = D_{++}(p) - e^{-\sigma p_0} D_{+-}(p) = \theta(p_0)\Delta_F(p) - \theta(-p_0)\Delta^*_F(p), \tag{8}
$$

which for bosons is $i/[p^2 - m^2 + i\varepsilon p_0]$, it is then of interest to consider transformations of Eq. (4) which could generate such linear combinations of the D_{ab}.

In this paper we will examine transformations to a $\widehat{D}_{XY}$ basis of the form

$$
U(p)D(p)U^T(-p) = \widehat{D}(p), \tag{9}
$$

where "T" denotes the transpose and

$$
U_{Xa}(p) = \begin{pmatrix} f_{1+}(p) & f_{1-}(p) \\ f_{2+}(p) & f_{2-}(p) \end{pmatrix}. \tag{10}
$$

Without further input, however, there would likely be no advantage to study such a transformation in general. One observation that could be a guide for choosing "useful" transformations is that since the original propagators $D_{ab}(p)$ satisfy the relations [2]

$$D_{++}(p) + D_{--}(p) - e^{-\sigma p_0} D_{+-}(p) - e^{\sigma p_0} D_{-+}(p) = 0, \tag{11}$$

$$D_{++}(p) + D_{--}(p) - \eta e^{\sigma p_0 - x} D_{-+}(p) - \eta e^{x - \sigma p_0} D_{+-}(p) = 0, \tag{12}$$

the transformed $\widehat{D}_{XY}(p)$ of Eq. (9) will also not be independent in general. It could then be of some advantage to choose the transformation so that either one or both of the relations of Eqs. (11, 12) translate into the trivial vanishing of either one or two of the $\widehat{D}_{XY}(p)$.

It may be noted that the transformation of Eq. (9) has a close correspondence to the Bogoliubov transformation studied in thermo field dynamics [14]. In fact, the transformation matrices $U(p)$ can be mapped onto a particular set of Bogoliubov transformations seeking a diagonal form for the propagator by a transformation of the field operators [5].

2 Transformed n–point functions

The first relation of Eq. (11) among the $D_{ab}(p)$ elements is an immediate consequence of the definition of the path-ordered product, whereas for the second one of Eq. (12) one also needs the Kubo-Martin-Schwinger (KMS) condition for equilibrium two–point correlation functions. These relations are thus also satisfied by the full propagator $G_{ab}(p)$. Hence, the full propagator has the same matrix components of Eq. (4) as the bare one, with now the Feynman propagators in Eq. (5) given by the spectral representation

$$\Delta_F(p) = \int_0^\infty d\tau^2 \frac{\rho(\tau, \vec{p})}{p_0^2 - \tau^2 + i\varepsilon}. \tag{13}$$

This implies that for the full propagator we may write the spectral form

$$G_{ab}(p) = \int_0^\infty d\tau^2 \rho(\tau, \vec{p}) D_{ab}(\tau, \vec{p}) \tag{14}$$

and the analogous one for the transformed propagator

$$\widehat{G}_{XY}(p) = \int_0^\infty d\tau^2 \rho(\tau, \vec{p}) \widehat{D}_{XY}(\tau, \vec{p}), \tag{15}$$

owing to the identical matrix structures of interacting and non-interacting propagators.

Consider now Dyson's equation for the full propagator defined on the contour $C_I \cup C_{II}$ [2, 9]:

$$\int_C dy\, \Gamma_C(x,y) G_C(y,z) = \delta_C(x-z), \tag{16}$$

or, in a matrix representation,

$$\int_{-\infty}^{+\infty} dy\, \Gamma(x,y)\sigma_3 G(y,z) = \sigma_3 \delta(x-z), \tag{17}$$

where $\sigma_3 = \mathrm{diag}(1,-1)$ is the 2×2 Pauli matrix. Writing $\Gamma_C(x,y) = \Gamma_C^{(0)}(x,y) - \Sigma_C(x,y)$, in momentum space we then have

$$G(p) = D(p) + D(p)\Sigma(p)G(p), \tag{18}$$

where here, and also in the following for n–point functions, the effects of σ_3 have been absorbed into the vertex function by associating a minus sign with each "$-$" index. From Eq. (9) and the preceding discussion we thus see

$$U(p)G(p)U^T(-p) = \widehat{G}(p), \tag{19}$$

where $\widehat{G}(p)$ satisfies

$$\widehat{G}(p) = \widehat{D}(p) + \widehat{D}(p)\widehat{\Sigma}(p)\widehat{G}(p) \tag{20}$$

and the transformed (amputated) self-energy $\widehat{\Sigma}(p)$ is given by

$$V(p)\Sigma(p)V^T(-p) = \widehat{\Sigma}(p), \tag{21}$$

where we have defined

$$V_{Xa}(p) = \begin{pmatrix} V_{1+}(p) & V_{1-}(p) \\ V_{2+}(p) & V_{2-}(p) \end{pmatrix} \equiv \left[U^T(-p)\right]^{-1}. \tag{22}$$

Generalizing this we can, as with Aurenche and Becherrawy [15, 16], introduce transformed (amputated) n–point functions $\widehat{\Gamma}_{X_1\cdots X_n}(p_1,\ldots,p_n)$ in terms of the $+/-$ functions $\Gamma_{a_1\cdots a_n}(p_1,\ldots,p_n)$. If all momenta are incoming, such functions are

$$\widehat{\Gamma}_{X_1\cdots\cdots X_n}(p_1,\ldots,p_n) = V_{X_1a_1}(p_1)\cdots V_{X_na_n}(p_n)\Gamma_{a_1\cdots a_n}(p_1,\ldots,p_n). \tag{23}$$

If some of the momenta were outgoing, say $p_{k+1}, \ldots, p_n$, the matrix $V^T(-p)$ rather than $V(p)$ would be associated with these momenta in Eq. (23). Because of this, the corresponding transformed n–point function is then simply related to that with all momenta incoming by

$$\widehat{\Gamma}_{X_1\cdots X_k;X_{k+1}\cdots X_n}(p_1,\ldots,p_k;p_{k+1},\ldots,p_n) = \widehat{\Gamma}_{X_1\cdots X_n}(p_1,\ldots,p_k,-p_{k+1},\ldots,-p_n). \tag{24}$$

Thus, it suffices to study these functions with all momenta incoming. We will assume this in the following, so that $\sum p_{i_0} = 0 = \sum \mu_i$, and we also use fermion number conservation: $\prod \eta_i = 1$. Also in what follows the delta functions associated with these conservation laws have been factored out from the Green functions.

Feynman rules for the transformed functions of Eq. (23) then follow from the rules for the n–point functions $\Gamma_{a_1\cdots a_n}$, which involve internal propagators D_{ab} connected by appropriate bare vertices $g_{ab\cdots}$:

$$g_{+\cdots+}(p_1,\ldots,p_n) = -g_{-\cdots-}(p_1,\ldots,p_n) \equiv g(p_1,\ldots,p_n), \tag{25}$$

with all others being zero [2]. Instead of these vertices one may also consider amputated subdiagrams, for which the following discussion will also apply. The internal propagators in a graph could be written in terms of transformed propagators $\widehat{D}_{XY}$ using Eq. (9) and, as also done by Aurenche and Becherrawy [15], the matrices of each such transformation could be associated with the adjacent vertices. This will ultimately replace each vertex $g_{ab\cdots}$ in a graph by the appropriate transformed vertex $\widehat{g}_{XY\cdots}$, where the transformation is given in Eq. (23). The end result is that the Feynman rules for the transformed n–point functions can be written solely in terms of transformed vertices connected by transformed internal propagators.

A simple illustration of this transformation of Feynman rules is a one-loop self-energy graph involving cubic interactions such as

$$-i\Sigma_{aa'}(p) = (-i)^2 \int \frac{d^4k}{(2\pi)^4}\, g_{abc}(p,-k,q) D_{bb'}(k) g_{a'b'c'}(-p,k,-q) D_{c'c}(q), \tag{26}$$

where $q = k - p$ and only thermal indices are retained. The transformed self-energy $\widehat{\Sigma}(p)$ of Eq. (21) corresponding to this graph is, using Eq. (9),

$$\begin{aligned} -i\widehat{\Sigma}_{XX'}(p) &= V_{Xa}(p)\,[-i\Sigma_{aa'}(p)]\,V_{X'a'}(-p) = \\ &= (-i)^2 \int \frac{d^4k}{(2\pi)^4}\, \widehat{g}_{XYZ}(p,-k,q)\widehat{D}_{YY'}(k)\widehat{g}_{X'Y'Z'}(-p,k,-q)\widehat{D}_{Z'Z}(q), \end{aligned} \tag{27}$$

where, as in Eq. (23),

$$\widehat{g}_{XYZ}(p,-k,q) = V_{Xa}(p)V_{Yb}(-k)V_{Zc}(q)g_{abc}(p,-k,q),$$
$$\widehat{g}_{X'Y'Z'}(-p,k,-q) = V_{X'a'}(-p)V_{Y'b'}(k)V_{Z'c'}(-q)g_{a'b'c'}(-p,k,-q). \quad (28)$$

It will be found convenient in the construction of the following transformations to aim for the $+/-$ n–point functions $\Gamma_{a_1\cdots a_n}$ to be multiplied by one of two particular factors for each index a_i with the value "$-$":

$$\Gamma_{a_1\cdots a_n}(p_1,\ldots,p_n) \prod_{a_i=-} e^{\sigma p_{i_0}},$$
$$\Gamma_{a_1\cdots a_n}(p_1,\ldots,p_n) \prod_{a_i=-} \eta_i e^{\sigma p_{i_0}-x_i}. \quad (29)$$

The exponential factors exactly eliminate the σ–dependence of $\Gamma_{a_1\cdots a_n}$, leaving these combinations σ–independent. The first function here is simply the $\sigma = 0$ n–point function expressed in terms of $\Gamma_{a_1\cdots a_n}$ with an arbitrary σ. The second function can be related to the first by the KMS relation. These functions satisfy

$$\sum_{a_i=\pm}\left[\Gamma_{a_1\cdots a_n}(p_1,\ldots,p_n) \prod_{a_i=-} e^{\sigma p_{i_0}}\right] = 0,$$
$$\sum_{a_i=\pm}\left[\Gamma_{a_1\cdots a_n}(p_1,\ldots,p_n) \prod_{a_i=-} \eta_i e^{\sigma p_{i_0}-x_i}\right] = 0, \quad (30)$$

which can be proven, for example, using the largest and smallest time equations of 't Hooft and Veltman [17, 18]. These relations are the amputated versions of the n–point generalizations of the 2–point relations of Eqs. (11, 12). Note that, unlike Eqs. (11, 12), in Eq. (30) no minus signs appear due to the their absorption into the amputated vertex functions, as discussed after Eq. (18).

We shall also have occasion to consider the particular combinations

$$\Gamma^{(R)}(p_1;p_2,\ldots,p_n) \equiv \sum_{a_i=\pm}\left[\Gamma_{+a_2\cdots a_n}(p_1,\ldots,p_n) \prod_{a_i=-} e^{\sigma p_{i_0}}\right],$$
$$\Gamma^{(A)}(p_1;p_2,\ldots,p_n) \equiv \sum_{a_i=\pm}\left[\Gamma_{+a_2\cdots a_n}(p_1,\ldots,p_n) \prod_{a_i=-} \eta_i e^{\sigma p_{i_0}-x_i}\right], \quad (31)$$

which are real-time representations of the one-particle-irreducible contributions to the generalized n–point retarded and advanced products [9, 16, 19]. They are related by complex conjugation: $\Gamma^{(A)}(p_1; p_2, \ldots, p_n) = \Gamma^{(R)\,*}(p_1; p_2, , \ldots, p_n)$, which can be shown using Eq. (30) together with the conjugation relation [16]

$$\Gamma^*_{a_1 \cdots a_n}(p_1, \ldots, p_n) = -\Gamma_{\bar{a}_1 \cdots \bar{a}_n}(p_1, \ldots, p_n) \prod_{a_i = -} \eta_i e^{x_i - 2\sigma p_{i_0}}, \tag{32}$$

where $\bar{a} = -, +$ when $a = +, -$. As will be seen later, the identification of $\Gamma^{(R)}$ and $\Gamma^{(A)}$ with the retarded and advanced amputated functions, respectively, is based on the convention in the imaginary-time formalism for the retarded and advanced analytic continuations of the (non-amputated) Green functions [10, 11, 12, 13].

3 The Keldysh basis

The Keldysh transformation of the time-path method is an example where one of the relations of Eqs. (11, 12) is used to eliminate one of the transformed functions $\widehat{D}_{XY}(p)$ [6–9]. Such a transformation is particularly useful in non-equilibrium situations, where a relation such as Eq. (12), which follows from the KMS condition, does not hold. We will find that this class of transformations automatically involves the retarded and advanced propagators $\Delta_R(p) = -\Delta^*_A(p)$ of Eq. (8), but there is some freedom in the form of the third non–zero element. As is conventional, we choose this element to be proportional to the combination $D_{++}(p) + D_{--}(p)$.

We consider the case where the transformed function $\widehat{D}_{22}(p)$ of Eq. (9) vanishes identically by Eq. (11). With $k_i = 1, 2$, the transformed propagators,

$$\widehat{D}_{k_1 k_2}(p) = \begin{pmatrix} \widehat{D}_{11}(p) & \widehat{D}_{12}(p) \\ \widehat{D}_{21}(p) & \widehat{D}_{22}(p) \end{pmatrix} = \begin{pmatrix} \Delta_S(p) & \Delta_R(p) \\ \Delta_A(p) & 0 \end{pmatrix}, \tag{33}$$

with the normalization on the right–hand–side, follow by choosing in Eq. (9)

$$U_{ka}(p) = \frac{1}{\sqrt{2} b(-p)} \begin{pmatrix} b(p)b(-p) & e^{\sigma p_0} b(p)b(-p) \\ 1 & -e^{\sigma p_0} \end{pmatrix}, \tag{34}$$

where $b(p)$ is an arbitrary function and the symmetric product $\Delta_S(p)$ is

$$\Delta_S(p) = b(p)b(-p)\left[D_{++}(p) + D_{--}(p)\right] = b(p)b(-p)\coth^{\eta}(x/2)\left[\Delta_R(p) - \Delta_A(p)\right]. \tag{35}$$

We then define "1/2" n–point functions $\mathcal{K}_{k_1\cdots k_n}(p_1,\ldots,p_n)$ in terms of the $+/-$ n–point functions $\Gamma_{a_1\cdots a_n}(p_1,\ldots,p_n)$ by Eq. (23). The corresponding matrix $V(p)$ of Eq. (22) in this case is, from Eq. (34),

$$\begin{pmatrix} V_{1+}(p) & V_{1-}(p) \\ V_{2+}(p) & V_{2-}(p) \end{pmatrix} = \frac{1}{\sqrt{2}b(-p)} \begin{pmatrix} 1 & e^{\sigma p_0} \\ b(p)b(-p) & -e^{\sigma p_0}b(p)b(-p) \end{pmatrix}. \quad (36)$$

In this notation the amputated 2–point functions are

$$\begin{aligned} &\mathcal{K}_{11}(p_1,p_2) = 0, \\ &\mathcal{K}_{12}(p_1,p_2) = \Gamma^{(A)}(p_1;p_2), \\ &\mathcal{K}_{21}(p_1,p_2) = \Gamma^{(R)}(p_1;p_2), \\ &\mathcal{K}_{22}(p_1,p_2) = b(p_1)b(p_2)\coth^{\eta}(x_{p_1}/2)\left[\Gamma^{(R)}(p_1;p_2) - \Gamma^{(A)}(p_1;p_2)\right], \end{aligned} \quad (37)$$

and in general one can show, using the largest/smallest time equation relations of Eq. (30), that

$$\begin{aligned} &\mathcal{K}_{11\cdots 1}(p_1,\ldots,p_n) = 0, \\ &\mathcal{K}_{211\cdots 1}(p_1,\ldots,p_n) = 2^{1-n/2}\frac{b(p_1)}{b(-p_2)\cdots b(-p_n)}\Gamma^{(R)}(p_1;p_2,\ldots,p_n), \end{aligned} \quad (38)$$

where $\Gamma^{(R)}(p_1;p_2,\ldots,p_n)$ is defined in Eq. (31).

Feynman rules for the 1/2 functions consist of internal propagators of Eq. (33) and bare vertices obtained from Eqs. (23, 36) using the bare $+/-$ vertices of Eq. (25):

$$\frac{\widehat{g}_{k_1\cdots k_n}(p_1,\ldots,p_n)}{g(p_1,\ldots,p_n)} = \frac{1-(-1)^{\#2}}{2^{n/2}}\prod_{k_i=1}\frac{1}{b(-p_i)}\prod_{k_i=2}b(p_i), \quad (39)$$

where "#2" denotes the number of "2" indices present. Thus, bare vertices with an even number of "2" indices vanish. The non-vanishing ones satisfy the following relation when the signs of all energies and momenta are reversed:

$$\frac{\widehat{g}_{k_1\cdots k_n}(-p_1,\ldots,-p_n)}{\widehat{g}_{k_1\cdots k_n}(p_1,\ldots,p_n)} = \frac{g(-p_1,\ldots,-p_n)}{g(p_1,\ldots,p_n)}\prod_{k_i=1,2}\frac{b(-p_i)}{b(p_i)}. \quad (40)$$

In these considerations there is no obvious advantage to choosing the free parameter $b(p)$ to be momentum dependent, and so choices such as $b(p)=1$ or $1/\sqrt{2}$ are usually most convenient.

As in this type of transformation only the one constraint of Eq. (11) satisfied by the D_{ab} was used to eliminate one of the transformed Green functions, the other constraint of Eq. (12) will lead to non-trivial relations amongst the transformed functions $\mathcal{K}_{k_1\cdots k_n}$. This is illustrated by the last relation of Eq. (37) relating $\mathcal{K}_{22}$ to $\mathcal{K}_{12}$ and $\mathcal{K}_{21}$, which is referred to as the fluctuation-dissipation theorem [9].

One could also consider the case where the transformed function $\widehat{D}_{22}(p)$ of Eq. (9) vanishes by Eq. (12), rather than by Eq. (11) as chosen here. The analysis of such a transformation is straightforward; we only note that in this case the transformed n–point function $\mathcal{K}_{211\cdots1}$ will be proportional to $\Gamma^{(A)}$, rather than to $\Gamma^{(R)}$ as happened in Eq. (38).

These Keldysh-like transformations are convenient in a general analysis of response theory [9]. In this approach one adds to the Hamiltonian a term involving a (weak) external current $J(t)$ and then calculates the response of various Green functions. Translating from the notation of Ref. [9] into that used here, one has

$$\begin{aligned}
\delta G_1(t) &\sim \int dt_1\; G_{12}(t,t_1)J(t_1) + \frac{1}{2}\int dt_1\, dt_2\; G_{122}(t,t_1,t_2)J(t_1)J(t_2) + \ldots, \\
\delta G_{11}(t,t') &\sim \int dt_1\; G_{112}(t,t',t_1)J(t_1) + \ldots,
\end{aligned} \tag{41}$$

and so on. The Green functions $G_{122\cdots2}(t_1,\ldots,t_n)$ with one "1" index are the n–point retarded products, and arise in the response of the expectation value $G_1(t)$ of the field operator, while those response functions with more than one "1" index arise in the response of higher-order correlation functions $G_{11\cdots1}$. The one-particle irreducible functions $\mathcal{K}_{k_1\cdots k_n}$ considered in this section enter into the construction of these response functions, and as such this setting provides an intuitive interpretation of them.

4 The RA basis

In equilibrium situations one can consider transformations in which both of the constraints of Eqs. (11, 12) translate into the trivial vanishing of two of the transformed functions $\widehat{D}_{XY}(p)$ of Eq. (9). This might be viewed as an attempt to "economize" the information contained in the original propagators $D_{ab}(p)$. We will consider two classes of such transformations. We first

look at the one that results by demanding $\widehat{D}_{11}(p)$ vanishes by Eq. (11) and $\widehat{D}_{22}(p)$ vanishes by Eq. (12). We find in this case that the transformation automatically leads to an off-diagonalization in terms of the retarded and advanced propagators [16]:

$$\widehat{D}_{\alpha_1\alpha_2}(p) = \begin{pmatrix} \widehat{D}_{RR}(p) & \widehat{D}_{RA}(p) \\ \widehat{D}_{AR}(p) & \widehat{D}_{AA}(p) \end{pmatrix} = \begin{pmatrix} 0 & \Delta_A(p) \\ \Delta_R(p) & 0 \end{pmatrix}, \tag{42}$$

where $\alpha_i = R$, A and, for the normalization on the right-hand-side to occur,

$$U_{\alpha a}(p) = \frac{1}{a(-p)} \begin{pmatrix} a(p)a(-p) & -e^{\sigma p_0} a(p)a(-p) \\ -\eta n(-x) & -\eta e^{\sigma p_0} n(x) \end{pmatrix}, \tag{43}$$

with $a(p)$ an arbitrary function.

By Eq. (23) we then define "RA" n–point functions $\mathcal{R}_{\alpha_1\cdots\alpha_n}(p_1,\ldots,p_n)$; the matrix $V(p)$ of Eq. (22) is, from Eq. (43),

$$\begin{pmatrix} V_{R+}(p) & V_{R-}(p) \\ V_{A+}(p) & V_{A-}(p) \end{pmatrix} = \frac{1}{a(-p)} \begin{pmatrix} -\eta n(-x) & \eta e^{\sigma p_0} n(x) \\ a(p)a(-p) & e^{\sigma p_0} a(p)a(-p) \end{pmatrix}. \tag{44}$$

In this notation the non-zero 2–point functions are the self-energies $\mathcal{R}_{RA}(p_1,p_2) = \mathcal{R}^*_{AR}(p_1,p_2) = \Gamma^{(R)}(p_1;p_2)$. Using the largest/smallest time equation relations of Eq. (30), one finds in general [15, 16]

$$\begin{aligned} &\mathcal{R}_{RR\cdots R}(p_1,\ldots,p_n) = 0 = \mathcal{R}_{AA\cdots A}(p_1,\ldots,p_n), \\ &\mathcal{R}_{RAA\cdots A}(p_1,\ldots,p_n) = \frac{a(p_2)\cdots a(p_n)}{a(-p_1)}\Gamma^{(R)}(p_1;p_2,\ldots,p_n), \\ &\mathcal{R}_{ARR\cdots R}(p_1,\ldots,p_n) = (-1)^n \frac{a(p_1)n(-x_{p_2})\cdots n(-x_{p_n})}{n(x_{p_1})a(-p_2)\cdots a(-p_n)}\Gamma^{(A)}(p_1;p_2,\ldots,p_n), \end{aligned} \tag{45}$$

where $\Gamma^{(R)}(p_1;p_2,\ldots,p_n)$ and $\Gamma^{(A)}(p_1;p_2,\ldots,p_n)$ appear in Eq. (31).

The RA functions satisfy a complex conjugation relation; relating $V_{\alpha a}(p)$ to $V_{\bar{\alpha}\bar{a}}(p)$ and using Eq. (32), one finds

$$\begin{aligned} \mathcal{R}^*_{\alpha_1\cdots\alpha_n}(p_1,p_2,\ldots,p_n) = -\mathcal{R}_{\bar{\alpha}_1\cdots\bar{\alpha}_n}(p_1,p_2,\ldots,p_n) \times \\ \prod_{\alpha_i=R} \frac{n(x_i)}{a(p_i)a(-p_i)} \prod_{\alpha_i=A} \frac{a(p_i)a(-p_i)}{-n(-x_i)}, \end{aligned} \tag{46}$$

where $\bar{\alpha} = A$, R for $\alpha = R$, A. This agrees with the results of Refs. [13, 16] for the choice $a(p) = -\eta n(x)$. We see by Eq. (46) that it is not possible to choose $a(p)$ so as to cancel the factors of the distribution function $n(x)$ arising in this complex conjugation relation.

Feynman rules for the RA functions consist of internal retarded or advanced propagators and bare vertices obtained from Eqs. (23, 44) using the bare $+/-$ vertices of Eq. (25):

$$\frac{\widehat{g}_{\alpha_1\cdots\alpha_n}(p_1,\ldots,p_n)}{g(p_1,\ldots,p_n)} = \prod_{\alpha_i=R} \frac{n(x_i)}{a(-p_i)} \prod_{\alpha_i=A} a(p_i) \left[\prod_{\alpha_i=R} e^{x_i} - \prod_{\alpha_i=R} \eta_i \right]. \qquad (47)$$

Thus, as for the full n–point functions in Eq. (45), bare vertices with all "R" or all "A" indices vanish: $\widehat{g}_{RR\cdots R} = 0 = \widehat{g}_{AA\cdots A}$. The non-vanishing ones satisfy

$$\frac{\widehat{g}_{\alpha_1\cdots\alpha_n}(-p_1,\ldots,-p_n)}{\widehat{g}_{\alpha_1\cdots\alpha_n}(p_1,\ldots,p_n)} = -(-1)^{\#R}\, \frac{g(-p_1,\ldots,-p_n)}{g(p_1,\ldots,p_n)} \prod_{\alpha_i=R,A} \frac{a(-p_i)}{a(p_i)}, \qquad (48)$$

where $\#R$ denotes the number of "R" indices present.

Two convenient choices of the free parameter $a(p)$ can be made based on these considerations. One is $a(p) = 1$, which normalizes the bare vertex $\widehat{g}_{RAA\cdots A}$ to the bare $+/-$ vertex g of Eq. (25). In this case the factor $\prod a(-p)/a(p)$ in Eq. (48) is unity, and the full n–point function $\mathcal{R}_{RAA\cdots A}$ of Eq. (45) is normalized to $\Gamma^{(R)}$. The other choice, as made in Refs. [15, 16], is $a(p) = -\eta n(x)$, which normalizes $\widehat{g}_{ARR\cdots R}$ to the bare $+/-$ vertex g. For this choice the factor $\prod a(-p)/a(p)$ in Eq. (48) is $(-1)^n$, and the full n–point function $\mathcal{R}_{ARR\cdots R}$ of Eq. (45) is normalized to $\Gamma^{(A)}$.

5 The $F\bar{F}$ basis

As a second example of using Eqs. (11, 12) to eliminate two of the transformed functions $\widehat{D}_{XY}(p)$ in Eq. (9), we consider the case where the off-diagonal components $\widehat{D}_{12}(p)$ and $\widehat{D}_{21}(p)$ vanish. To avoid a singular transformation we find we have to impose, for example, that $\widehat{D}_{21}(p)$ vanishes by Eq. (11) for $p_0 > 0$ and by Eq. (12) for $p_0 < 0$, and *vice-versa* for $\widehat{D}_{12}(p)$. Such a transformation automatically leads to a diagonalization in terms of the

Feynman and anti-Feynman propagators:

$$\widehat{D}_{f_1 f_2}(p) = \begin{pmatrix} \widehat{D}_{FF}(p) & \widehat{D}_{F\bar{F}}(p) \\ \widehat{D}_{\bar{F}F}(p) & \widehat{D}_{\bar{F}\bar{F}}(p) \end{pmatrix} = \begin{pmatrix} \Delta_F(p) & 0 \\ 0 & \Delta_F^*(p) \end{pmatrix}, \tag{49}$$

where $f_i = F,\, \bar{F}$ and, for the normalization on the right-hand-side to occur,

$$\begin{aligned} U_{fa}(p) = \theta(p_0) & \begin{pmatrix} c(p) & -\eta e^{\sigma p_0 - x} c(p) \\ -\eta d(p) & \eta e^{\sigma p_0} d(p) \end{pmatrix} \\ + \theta(-p_0) & \frac{n(-x) e^{\sigma p_0 - x}}{c(-p) d(-p)} \begin{pmatrix} e^{-\sigma p_0} d(-p) & -d(-p) \\ -e^{x - \sigma p_0} c(-p) & \eta c(-p) \end{pmatrix}, \end{aligned} \tag{50}$$

where $c(p)$ and $d(p)$ are arbitrary functions.

We next introduce "$F\bar{F}$" n–point functions $\mathcal{F}_{f_1 \cdots f_n}(p_1, \ldots, p_n)$ by Eq. (23); the matrix $V(p)$ of Eq. (22) is, from Eq. (50),

$$\begin{aligned} \begin{pmatrix} V_{F+}(p) & V_{F-}(p) \\ V_{\bar{F}+}(p) & V_{\bar{F}-}(p) \end{pmatrix} = \theta(p_0) & \begin{pmatrix} c(p) & \eta e^{\sigma p_0 - x} c(p) \\ \eta d(p) & \eta e^{\sigma p_0} d(p) \end{pmatrix} \\ + \theta(-p_0) & \frac{n(-x) e^{\sigma p_0 - x}}{c(-p) d(-p)} \begin{pmatrix} e^{-\sigma p_0} d(-p) & d(-p) \\ e^{x - \sigma p_0} c(-p) & \eta c(-p) \end{pmatrix}. \end{aligned} \tag{51}$$

The non-vanishing 2–point functions are then $\mathcal{F}_{FF}(p_1, p_2) = -\mathcal{F}^*_{\bar{F}\bar{F}}(p_1, p_2) = \theta(p_{1_0}) \Gamma^{(R)}(p_1; p_2) + \theta(-p_{1_0}) \Gamma^{(A)}(p_1; p_2)$. It is worth noting that $\mathcal{F}_{FF}$, not the original amputated 2–point function Γ_{++}, determines the pole of the time-ordered Green function $G_{++}(p)$ of Eq. (9), and that at finite temperature these two functions do not coincide, even at the one–loop level [2, 18]. A similar lack of equivalence can also be shown for the higher n–point functions $\mathcal{F}_{FF \cdots F}$ and $\Gamma_{++\cdots+}$.

Feynman rules for the $F\bar{F}$ functions consist of the Feynman and anti-Feynman propagators and bare $F\bar{F}$ vertices obtained from Eqs. (23, 51) using the bare $+/-$ vertices of Eq. (25):

$$\begin{aligned} & \widehat{g}_{f_1 \cdots f_n}(p_1, \ldots, p_n) / g(p_1, \ldots, p_n) = \\ & \prod_{f_i = F} \left[\theta(p_{i_0}) c(p_i) + \theta(-p_{i_0}) \frac{e^{-x_i} n(-x_i)}{c(-p_i)} \right] \prod_{f_i = \bar{F}} \left[\eta_i \theta(p_{i_0}) d(p_i) + \theta(-p_{i_0}) \frac{n(-x_i)}{d(-p_i)} \right] - \\ & \prod_{f_i = F} \left[\theta(p_{i_0}) c(p_i) + \eta_i \theta(-p_{i_0}) \frac{n(-x_i)}{c(-p_i)} \right] \prod_{f_i = \bar{F}} \left[\theta(p_{i_0}) e^{x_i} d(p_i) + \theta(-p_{i_0}) \frac{n(-x_i)}{d(-p_i)} \right] \end{aligned} \tag{52}$$

which satisfy

$$\frac{\widehat{g}_{f_1\cdots f_n}(-p_1,\ldots,-p_n)}{\widehat{g}_{\bar{f}_1\cdots\bar{f}_n}(p_1,\ldots,p_n)} = -\frac{g(-p_1,\ldots,-p_n)}{g(p_1,\ldots,p_n)} \times \prod_{f_i=F,\bar{F}} \left[\theta(p_{i_0})\frac{n(x_i)}{c(p_i)d(p_i)} + \theta(-p_{i_0})\frac{c(-p_i)d(-p_i)}{n(-x_i)}\right]. \tag{53}$$

Two considerations illustrate the relative effects of different choices of the arbitrary coefficients $c(p)$ and $d(p)$ in this basis. One is the normalization of the $F\bar{F}$ n–point functions, for which we have, for example,

$$\begin{aligned}
&\theta(k_0)\theta(-l_0)\cdots\theta(-r_0)\mathcal{F}_{FF\cdots F}(k,l,\ldots,r) = \\
&= \theta(k_0)\theta(-l_0)\cdots\theta(-r_0)\left[\frac{c(k)n(-x_l)\cdots n(-x_r)}{n(x_k)c(-l)\cdots c(-r)}\right]\Gamma^{(R)}(k;l,\ldots,r), \\
&\theta(k_0)\theta(-l_0)\cdots\theta(-q_0)\theta(r_0)\mathcal{F}_{FF\cdots F\bar{F}}(k,l,\ldots,q,r) = \\
&= \theta(k_0)\theta(-l_0)\cdots\theta(-q_0)\theta(r_0)\left[\frac{c(k)n(-x_l)\cdots n(-x_q)d(r)}{n(x_k)c(-l)\cdots c(-q)\eta_r e^{-x_r}}\right]\Gamma^{(R)}(k;l,\ldots,q,r), \\
&\theta(-k_0)\theta(l_0)\cdots\theta(r_0)\mathcal{F}_{\bar{F}\bar{F}\cdots\bar{F}}(k,l,\ldots,r) = \\
&= -\theta(-k_0)\theta(l_0)\cdots\theta(r_0)\left[\frac{d(l)\cdots d(r)}{d(-k)}\right]\Gamma^{(R)}(k;l,\ldots,r), \\
&\theta(-k_0)\theta(l_0)\cdots\theta(r_0)\mathcal{F}_{FF\cdots F}(k,l,\ldots,r) = \\
&= \theta(-k_0)\theta(l_0)\cdots\theta(r_0)\left[\frac{c(l)\cdots c(r)}{c(-k)}\right]\Gamma^{(A)}(k;l,\ldots,r), \\
&\theta(-k_0)\theta(l_0)\cdots\theta(q_0)\theta(-r_0)\mathcal{F}_{FF\cdots F\bar{F}}(k,l,\ldots,q,r) = \\
&= \theta(-k_0)\theta(l_0)\cdots\theta(q_0)\theta(-r_0)\left[\frac{c(l)\cdots c(q)n(-x_r)}{c(-k)d(-r)}\right]\Gamma^{(A)}(k;l,\ldots,q,r), \\
&\theta(k_0)\theta(-l_0)\cdots\theta(-r_0)\mathcal{F}_{\bar{F}\bar{F}\cdots\bar{F}}(k,l,\ldots,r) = \\
&= -\theta(k_0)\theta(-l_0)\cdots\theta(-r_0)\left[\frac{d(k)n(-x_l)\cdots n(-x_r)}{n(x_k)d(-l)\cdots d(-r)}\right]\Gamma^{(A)}(k;l,\ldots,r),
\end{aligned} \tag{54}$$

where $\Gamma^{(R)}(k;l,\ldots,r)$ and $\Gamma^{(A)}(k;l,\ldots,r)$ are defined in Eq. (31). The other consideration is complex conjugation: relating V_{fa} to $V_{\bar{f}\bar{a}}$ from Eq. (51) and using Eq. (32) one can show

$$\mathcal{F}^*_{f_1\cdots f_n}(p_1,\ldots,p_n) = -\mathcal{F}_{\bar{f}_1\cdots\bar{f}_n}(p_1,\ldots,p_n) \times$$

$$\prod_{f_i=F}\left[\theta(p_{i_0})\frac{c(p_i)}{d(p_i)}+\theta(-p_{i_0})\frac{d(-p_i)}{c(-p_i)}\right]\prod_{f_i=\bar{F}}\left[\theta(p_{i_0})\frac{d(p_i)}{c(p_i)}+\theta(-p_{i_0})\frac{c(-p_i)}{d(-p_i)}\right]\eta_i e^{x_i}, \tag{55}$$

where $\bar{f}=\bar{F}$, F for $f=F$, $\bar{F}$.

We consider now in the preceding context two choices of the parameters $c(p)$ and $d(p)$. For the first one, we begin by noting that the "symmetrical" or "unitary" formulation of thermo field dynamics is recovered by the choices

$$\begin{aligned}\sigma &= \beta/2,\\ c(p) &= e^{x/2}\sqrt{n(x)},\\ d(p) &= \eta e^{-\beta\mu/2}\sqrt{n(x)},\end{aligned} \tag{56}$$

in the transformation matrix of Eq. (51) [2, 4, 14]. Of course, potential problems exist for massive bosons in this when taking the square root of the distribution function. However, the θ functions of energy which accompany $c(p)$ and $d(p)$ in general guarantee that this square root remains real except in the case of finite chemical potential with $0<(p_0/\mu)<1$, where a separate analysis in this and the following is required [2, 15]. Guided by the values of Eq. (56) we then consider, for arbitrary σ, the choices

$$\begin{aligned}c(p) &= e^{x/2}\sqrt{n(x)},\\ d(p) &= \eta\sqrt{n(x)}.\end{aligned} \tag{57}$$

In this case the bare vertices of Eq. (52) are

$$\frac{\widehat{g}_{f_1\cdots f_n}(p_1,\ldots,p_n)}{g(p_1,\ldots,p_n)}=\prod_{f_i=F}C[p_i]\prod_{f_i=\bar{F}}\eta_i S[p_i]-\prod_{f_i=F}S[p_i]\prod_{f_i=\bar{F}}C[p_i], \tag{58}$$

which satisfy

$$\frac{\widehat{g}_{f_1\cdots f_n}(-p_1,\ldots,-p_n)}{\widehat{g}_{\bar{f}_1\cdots\bar{f}_n}(p_1,\ldots,p_n)}=-\frac{g(-p_1,\ldots,-p_n)}{g(p_1,\ldots,p_n)}, \tag{59}$$

and where

$$\begin{aligned}C[p] &= \theta(p_0)e^{x/2}\sqrt{n(x)}+\theta(-p_0)e^{-x/2}\sqrt{n(-x)},\\ S[p] &= \theta(p_0)\sqrt{n(x)}+\eta\theta(-p_0)\sqrt{n(-x)},\end{aligned} \tag{60}$$

which obey $C^2[p] - \eta S^2[p] = 1$. This choice of parameters leads to the following prefactors in the normalization relations of Eq. (54):

$$\begin{aligned}
&\frac{c(k)n(-x_l)\cdots n(-x_r)}{n(x_k)c(-l)\cdots c(-r)} = \sqrt{\frac{n(-x_l)\cdots n(-x_r)}{n(x_k)}},\\
&\frac{c(k)n(-x_l)\cdots n(-x_q)d(r)}{n(x_k)c(-l)\cdots c(-q)\eta_r e^{-x_r}} = e^{x_r/2}\sqrt{\frac{n(-x_l)\cdots n(-x_q)n(x_r)}{n(x_k)}},\\
&\frac{d(l)\cdots d(r)}{d(-k)} = \sqrt{\frac{n(x_l)\cdots n(x_r)}{n(-x_k)}},\\
&\frac{c(l)\cdots c(r)}{c(-k)} = \sqrt{\frac{n(x_l)\cdots n(x_r)}{n(-x_k)}},\\
&\frac{c(l)\cdots c(q)n(-x_r)}{c(-k)d(-r)} = \eta_r e^{-x_r/2}\sqrt{\frac{n(x_l)\cdots n(x_q)n(-x_r)}{n(-x_k)}},\\
&\frac{d(k)n(-x_l)\cdots n(-x_r)}{n(x_k)d(-l)\cdots d(-r)} = \sqrt{\frac{n(-x_l)\cdots n(-x_r)}{n(x_k)}},
\end{aligned} \tag{61}$$

while the complex conjugation relation of Eq. (55) is

$$\mathcal{F}^*_{f_1\cdots f_n}(k,\ldots,r) = -\mathcal{F}_{\bar{f}_1\cdots\bar{f}_n}(k,\ldots,r)\prod_{f_i=\bar{F}}\eta_i. \tag{62}$$

As a second choice of the parameters $c(p)$ and $d(p)$, we consider

$$\begin{aligned}
c(p) &= e^x n(x),\\
d(p) &= \eta.
\end{aligned} \tag{63}$$

This choice leads to bare vertices of Eq. (52) of the form

$$\begin{aligned}
\frac{\widehat{g}_{f_1\cdots f_n}(p_1,\ldots,p_n)}{g(p_1,\ldots,p_n)} &= \prod_{f_i=F}\left[1+\eta_i\theta(p_{i_0})n(x_i)\right]\prod_{f_i=\bar{F}}\left[\theta(p_{i_0})+\eta_i\theta(-p_{i_0})n(-x_i)\right] -\\
&\prod_{f_i=F}\left[\eta_i\theta(p_{i_0})n(x_i)+\theta(-p_{i_0})\right]\prod_{f_i=\bar{F}}\left[1+\eta_i\theta(-p_{i_0})n(-x_i)\right],
\end{aligned} \tag{64}$$

which satisfy

$$\frac{\widehat{g}_{f_1\cdots f_n}(-p_1,\ldots,-p_n)}{\widehat{g}_{\bar{f}_1\cdots\bar{f}_n}(p_1,\ldots,p_n)} = -\frac{g(-p_1,\ldots,-p_n)}{g(p_1,\ldots,p_n)}. \tag{65}$$

In this case the prefactors in the normalization conditions of Eq. (54) become

$$\begin{aligned}
&\frac{c(k)n(-x_l)\cdots n(-x_r)}{n(x_k)c(-l)\cdots c(-r)} = 1, \\
&\frac{c(k)n(-x_l)\cdots n(-x_q)d(r)}{n(x_k)c(-l)\cdots c(-q)\eta_r e^{-x_r}} = 1, \\
&\frac{d(l)\cdots d(r)}{d(-k)} = 1, \\
&\frac{c(l)\cdots c(r)}{c(-k)} = \frac{n(x_l)\cdots n(x_r)}{n(-x_k)}, \\
&\frac{c(l)\cdots c(q)n(-x_r)}{c(-k)d(-r)} = \eta_r e^{-x_r}\frac{n(x_l)\cdots n(x_q)n(-x_r)}{n(-x_k)}, \\
&\frac{d(k)n(-x_l)\cdots n(-x_r)}{n(x_k)d(-l)\cdots d(-r)} = \frac{n(-x_l)\cdots n(-x_r)}{n(x_k)},
\end{aligned} \tag{66}$$

while the complex conjugation relation of Eq. (55) is

$$\begin{aligned}
&\mathcal{F}^*_{f_1\cdots f_n}(p_1,\ldots,p_n) = -\mathcal{F}_{\bar{f}_1\cdots\bar{f}_n}(p_1,\ldots,p_n)\times \\
&\prod_{f_i=F}\left[\theta(p_{i_0})n(x_i)+\frac{\theta(-p_{i_0})}{n(-x_i)}\right]\prod_{f_i=\bar{F}}\left[\frac{\theta(p_{i_0})}{n(x_i)}+\theta(-p_{i_0})n(-x_i)\right]\eta_i e^{-x_i}
\end{aligned} \tag{67}$$

One can also consider parameters $c(p)$ and $d(p)$ that represent choices that interpolate between the two competing features of normalization and complex conjugation. For example, the choice $c(p) = e^{x/2}$ and $d(p) = 1$ leads to a simple complex conjugation relation of Eq. (55) but also to a somewhat more involved set of normalization relations of Eq. (54); this has the advantage that no square roots of distribution functions need be taken, but at the expense of having a more complicated form of Eq. (53) relating the bare vertex $\widehat{g}_{f_1\cdots f_n}(-p_1,\ldots,-p_n)$ to $\widehat{g}_{\bar{f}_1\cdots\bar{f}_n}(p_1,\ldots,p_n)$.

6 Comparisons of the bases

In this section we look at the transformations between the three bases considered here. As well as comparing their respective structures, this will allow us to see how relations in one basis translate into relations in another basis. We first consider the transformation between the RA functions and the

Keldysh-like 1/2 functions. Writing this relation as

$$\mathcal{R}_{\alpha_1\cdots\alpha_n}(p_1,\ldots,p_n) = T_{\alpha_1 k_1}(p_1)\cdots T_{\alpha_n k_n}(p_n)\mathcal{K}_{k_1\cdots k_n}(p_1,\ldots,p_n) \tag{68}$$

we find, for the choice $b(p) = 1/\sqrt{2}$ and $a(p) = 1$ in Eqs. (36, 44), the matrix

$$\begin{pmatrix} T_{R1}(p) & T_{R2}(p) \\ T_{A1}(p) & T_{A2}(p) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\coth^{\eta}(x/2) & 1 \\ 1 & 0 \end{pmatrix}. \tag{69}$$

With this transformation, it is seen that the identity $\mathcal{R}_{AA\cdots A} = 0$ in the RA basis translates in the Keldysh basis to $\mathcal{K}_{11\cdots 1} = 0$. The other identity in the RA basis, $\mathcal{R}_{RR\cdots R} = 0$, gives a relation amongst various $\mathcal{K}_{k_1\cdots k_n}$ functions. For the 2–point function this relation is the fluctuation-dissipation theorem of the last relation of Eq. (37), while for higher n–point functions such a relation can be used to construct what have been advocated as generalizations of the fluctuation-dissipation theorem [9]. One can also see using Eq. (69) that the n–point retarded functions in the two bases are related as $\mathcal{R}_{RAA\cdots A} = \mathcal{K}_{211\cdots 1}$.

It is also of interest to examine other "mixed" functions, as for example

$$\begin{aligned}\mathcal{R}_{RRAA\cdots A}(p_1,p_2,\ldots,p_n) &= \mathcal{K}_{2211\cdots 1}(p_1,p_2,\ldots,p_n) + \\ &+\frac{1}{2}\coth^{\eta}(x_{p_1}/2)\mathcal{K}_{1211\cdots 1}(p_1,p_2,\ldots,p_n) + \\ &+\frac{1}{2}\coth^{\eta}(x_{p_2}/2)\mathcal{K}_{2111\cdots 1}(p_1,p_2,\ldots,p_n).\end{aligned} \tag{70}$$

Consider first the 3–point functions. Recent studies have examined the relation between n–point functions of real-time formalisms and the analytically continued functions $\Gamma^{(n)}(z_1,\ldots,z_n)$ of the imaginary-time formalism, where initially all energy components of z_i are in the discrete set of Matsubara frequencies on the imaginary axis. For the retarded products the relation is [10, 11, 12, 13, 15]

$$\begin{aligned}\mathcal{R}_{RAA\cdots A}(p_1,p_2,\ldots,p_n) &= \mathcal{K}_{211\cdots 1}(p_1,p_2,\ldots,p_n) = \\ &= \Gamma^{(n)}(z_1\to p_{1_0}+i(n-1)\varepsilon, z_2\to p_{2_0}-i\varepsilon,\ldots,z_n\to p_{n_0}-i\varepsilon).\end{aligned} \tag{71}$$

Together with the complex conjugation relation of Eq. (46), this allows us to infer from Eq. (70) the relation between the 3–point response function $\mathcal{K}_{221}$ and various analytic continuations of the imaginary-time functions. Such a relation to the imaginary-time functions can also be worked out using

Eq. (70) for the 4–point response function $\mathcal{K}_{2211}$. For this we need, as well as Eq. (71), the relation between, for example, the mixed RA function $\mathcal{R}_{AARR}$ and various analytic continuations of the 4–point imaginary-time function [13, 20].

We also note that Eq. (70) can provide an interpretation of the "mixed" RA functions in terms of response functions in the Keldysh basis. As a result, certain causal properties (in time) can be inferred. For example, since the response relations of Eq. (41) are causal in nature, we conclude that each of the functions $G_{1222\cdots 2}(t_1, t_2, \ldots)$, $G_{2122\cdots 2}(t_1, t_2, \ldots)$, and $G_{1122\cdots 2}(t_1, t_2, \ldots)$ should only have t_1 or t_2 as the largest time [9, 19]. By a similar analysis for the non-amputated mixed functions $G_{RRAA\cdots A}$ as that leading to Eq. (70), one could then deduce the corresponding causal properties for these mixed RA Green functions.

We next consider the transformation between the $F\bar{F}$ functions and those of the RA basis. Writing this as

$$\mathcal{R}_{\alpha_1\cdots\alpha_n}(p_1, \ldots, p_n) = T_{\alpha_1 f_1}(p_1)\cdots T_{\alpha_n f_n}(p_n)\mathcal{F}_{f_1\cdots f_n}(p_1, \ldots, p_n) \tag{72}$$

we find, for the choices $a(p) = 1$ and $c(p) = e^x n(x)$, $d(p) = \eta$ in Eqs. (44, 51), the matrix

$$\begin{pmatrix} T_{RF}(p) & T_{R\bar{F}}(p) \\ T_{AF}(p) & T_{A\bar{F}}(p) \end{pmatrix} = \theta(p_0)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \theta(-p_0)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{73}$$

From this one sees that an F index corresponds to an R index for positive energies and to an A index for negative energies, and *vice-versa* for an $\bar{F}$ index. As well, with this relation one can translate the identities $\mathcal{R}_{RR\cdots R} = 0 = \mathcal{R}_{AA\cdots A}$ of the RA basis into the following set of relations in the $F\bar{F}$ basis:

$$\begin{aligned} &\theta(k_0)\cdots\theta(m_0)\theta(-n_0)\cdots\theta(-r_0)\mathcal{F}_{F\cdots F\,\bar{F}\cdots\bar{F}}(k, \ldots, m, n, \ldots, r) = 0, \\ &\theta(-k_0)\cdots\theta(-m_0)\theta(n_0)\cdots\theta(r_0)\mathcal{F}_{F\cdots F\,\bar{F}\cdots\bar{F}}(k, \ldots, m, n, \ldots, r) = 0. \end{aligned} \tag{74}$$

These relations have a direct causality interpretion: if an F index represents a particle ($p_0 > 0$) travelling forward in time and an antiparticle ($p_0 < 0$) travelling backwards in time, and an $\bar{F}$ index represents the time-reversed situation, then Eq. (74), together with the complex conjugation relation of Eq. (55), states that one cannot have particles and antiparticles either all

disappearing into the vacuum nor all appearing out of the vacuum. Such an interpretation also holds for arbitrary choices of the parameters $a(p)$, $c(p)$, and $d(p)$.

With the inverse of the transformation of Eq. (73) we can, given relations such as Eq. (71) relating the RA functions to the analytically continued imaginary-time functions $\Gamma^{(n)}$, examine how the $F\bar{F}$ functions are related to the imaginary-time functions. To simplify the notation, from now on we omit from $\Gamma^{(n)}(z_1, \ldots, z_n)$ the last complex argument z_n, which by energy-momentum conservation is minus the sum of the others. We first consider, with $p + q = 0$, the 2–point function $\mathcal{F}_{FF}(p, q) = -\mathcal{F}^*_{\bar{F}\bar{F}}(p, q)$, for which we find

$$\begin{aligned}\mathcal{F}_{FF}(p,q) &= \theta(p_0)\Gamma^{(R)}(p;q) + \theta(-p_0)\Gamma^{(A)}(p;q) = \\ &= \theta(p_0)\Gamma^{(2)}(z \to p_0 + i\varepsilon) + \theta(-p_0)\Gamma^{(2)}(z \to p_0 - i\varepsilon) = \Gamma^{(2)}(z \to p_0 + i\varepsilon p_0),\end{aligned} \tag{75}$$

as expected [2]. The other two 2–point functions $\mathcal{F}_{F\bar{F}}(p, q) \sim \mathcal{F}^*_{\bar{F}F}(p, q)$ vanish identically, as can be shown by multiplying $\mathcal{F}_{F\bar{F}}(p, q)$ by $1 = \theta(p_0) + \theta(-p_0)$ and then using the causality relations of Eq. (74).

We next consider the 3–point function. With $p + q + r = 0$, we first examine $\mathcal{F}_{FFF}(p, q, r)$, which using Eqs. (46, 71, 73) can be written as

$$\begin{aligned}\mathcal{F}_{FFF}(p,q,r) &= \theta(-p_0)\theta(-q_0)\Gamma^{(3)}(z_1 \to p_0 - i\varepsilon, z_2 \to q_0 - i\varepsilon) \\ &+\theta(-p_0)\theta(-r_0)\Gamma^{(3)}(z_1 \to p_0 - i\varepsilon, z_2 \to q_0 + 2i\varepsilon) \\ &+\theta(-q_0)\theta(-r_0)\Gamma^{(3)}(z_1 \to p_0 + 2i\varepsilon, z_2 \to q_0 - i\varepsilon) \\ &+\theta(p_0)\theta(q_0)\frac{n(x_p)n(x_q)}{n(-x_r)}\Gamma^{(3)}(z_1 \to p_0 + i\varepsilon, z_2 \to q_0 + i\varepsilon) \\ &+\theta(p_0)\theta(r_0)\frac{n(x_p)n(x_r)}{n(-x_q)}\Gamma^{(3)}(z_1 \to p_0 + i\varepsilon, z_2 \to q_0 - 2i\varepsilon) \\ &+\theta(q_0)\theta(r_0)\frac{n(x_q)n(x_r)}{n(-x_p)}\Gamma^{(3)}(z_1 \to p_0 - 2i\varepsilon, z_2 \to q_0 + i\varepsilon).\end{aligned} \tag{76}$$

In this form $\mathcal{F}_{FFF}$ appears to be a linear combination of the three possible analytic continuations of the 3–point imaginary-time function and their complex conjugates, but the presence of the θ functions allows the replacement in Eq. (76) of all the $\Gamma^{(3)}(z_1, z_2)$ functions by a single analytically continued

function, $\Gamma^{(3)}(z_1 \to p_0 + i\varepsilon p_0, z_2 \to q_0 + i\varepsilon q_0)$ [21, 22]. Up to a normalization factor, then, this result could have been anticipated from the 2–point function $\mathcal{F}_{FF}(p,q)$ of Eq. (75). In fact, a similar analysis for the n–point function $\mathcal{F}_{F\cdots F}(p_1,\ldots,p_n)$ indicates that this "Feynman" function in general can be written in terms of a single analytically continued imaginary-time function with $z_i \to p_{i_0} + i\varepsilon p_{i_0}$.

One can also analyze the "mixed" $F\bar{F}$ functions in a similar way. For example, for the 3–point function $\mathcal{F}_{FF\bar{F}}(p,q,r)$ we find

$$\begin{aligned}
\mathcal{F}_{FF\bar{F}}(p,q,r) &= \theta(q_0)\theta(r_0)\Gamma^{(3)}(z_1 \to p_0 - i\varepsilon, z_2 \to q_0 + 2i\varepsilon) \\
&\quad +\theta(p_0)\theta(r_0)\Gamma^{(3)}(z_1 \to p_0 + 2i\varepsilon, z_2 \to q_0 - i\varepsilon) \\
&\quad -\theta(-q_0)\theta(-r_0)\frac{n(x_p)n(x_r)}{n(-x_q)}\Gamma^{(3)}(z_1 \to p_0 + i\varepsilon, z_2 \to q_0 - 2i\varepsilon) \\
&\quad -\theta(-p_0)\theta(-r_0)\frac{n(x_q)n(x_r)}{n(-x_p)}\Gamma^{(3)}(z_1 \to p_0 - 2i\varepsilon, z_2 \to q_0 + i\varepsilon). \quad (77)
\end{aligned}$$

However, unlike the "pure" Feynman n–point function $\mathcal{F}_{FF\cdots F}$, there does not appear to be any simple interpretation of these mixed $F\bar{F}$ functions in terms of single analytic continuations of the imaginary-time functions.

Let us compare the number of independent functions present in all three bases. Taking into account the complex conjugation relation of Eq. (46) in the RA basis and Eq. (55) in the $F\bar{F}$ basis, as well as Eq. (37) for the Keldysh basis, all three bases are seen to contain one independent 2–point function, which can be taken as the retarded function $\mathcal{R}_{RA} = \mathcal{K}_{21}$ or the Feynman function $\mathcal{F}_{FF}$. For the 3–point function the RA basis has three independent functions, which could be taken as the three retarded functions $\mathcal{R}_{RAA}$, $\mathcal{R}_{ARA}$, and $\mathcal{R}_{AAR}$. The Keldysh basis contains these three retarded functions – $\mathcal{K}_{211}$, $\mathcal{K}_{121}$, and $\mathcal{K}_{112}$ – but also four additional functions: $\mathcal{K}_{222}$, $\mathcal{K}_{122}$, $\mathcal{K}_{212}$, and $\mathcal{K}_{221}$. However, relations such as Eq. (70) and that following from $\mathcal{R}_{RRR} = 0$ in Eq. (69) shows that these latter four can be written as linear combinations of the retarded functions and their complex conjugates. Similarly, taking into account complex conjugation, the $F\bar{F}$ basis appears to have four 3–point functions, which could be taken as $\mathcal{F}_{FFF}$, $\mathcal{F}_{\bar{F}FF}$, $\mathcal{F}_{F\bar{F}F}$, and $\mathcal{F}_{FF\bar{F}}$. However, relations such as Eqs. (76, 77) show that all four can be written in terms of the three retarded products and their complex conjugates. Alternatively, recalling that these "mixed" $F\bar{F}$ functions are constrained by the relations of Eq. (74) forcing energy to flow in only certain directions,

one could in this context count as an independent function each distinct orientation of the flow of external energy. In this way the RA basis has $3 \times 6 = 18$ such independent oriented functions, while the $F\bar{F}$ basis also has $1 \times 6 + 3 \times 4 = 18$ independent oriented functions. One thus concludes that all three bases contain the same number of independent 3–point functions.

Various examples could serve to compare the convenience of these bases in practical calculations. A simple and illustrative such example is the one-loop self-energy diagram of Eq. (27) with a bare $+/-$ coupling constant g of Eq. (25) independent of momentum. The corresponding example in the imaginary-time formalism using the retarded analytic continuation $z \to p_0 + i\varepsilon$ has been explored by Weldon [23]. Let us then find from Eq. (27) the retarded self-energy. We first consider the Keldysh and RA bases, for which we calculate $\mathcal{K}_{21}(p,-p) = \mathcal{R}_{RA}(p,-p)$. In the Keldysh basis the non-vanishing bare vertices needed are $\widehat{g}_{211}$ and permutations of its indices, since $\widehat{D}_{22}(p) = 0$, and so of all the possible terms in Eq. (27) only two survive. In the RA basis the relevant non-vanishing bare vertices are $\widehat{g}_{RAA}$ and $\widehat{g}_{ARR}$ and permutations of their indices, since $\widehat{D}_{RR}(p) = 0 = \widehat{D}_{AA}(p)$. Three possible contributions then remain in Eq. (27), but two of these disappear upon integration over k_0 because of the presence of terms like $\Delta_R(k)\Delta_R(q)$ or $\Delta_A(k)\Delta_A(q)$ with poles in the same half-plane [15]. In both bases one thus obtains, with $q = k - p$ and neglecting any γ-matrix structure,

$$\begin{aligned}
\mathcal{K}_{21}(p,-p) &= \mathcal{R}_{RA}(p,-p) = \\
&= -ig^2 \int \frac{d^4k}{(2\pi)^4} \Big\{ \Delta_R(k) \left[\frac{1}{2} + \eta_q n(x_q) \right] [\Delta_R(q) - \Delta_A(q)] + \\
&\qquad + \Delta_A(q) \left[\frac{1}{2} + \eta_k n(x_k) \right] [\Delta_R(k) - \Delta_A(k)] \Big\} , \qquad (78)
\end{aligned}$$

which agrees with previous results [15, 23, 24].

In this calculation the choice of free parameters $b(p) = 1$ or $1/\sqrt{2}$ in the Keldysh basis and $a(p) = 1$ or $-\eta n(x)$ in the RA basis are easiest to use. The form of the result of Eq. (78) is convenient for doing the k_0 integration, since for free propagators $\Delta_R(k) - \Delta_A(k) = 2\pi\varepsilon(k_0)\delta(k^2 - m^2)$. Such a form also arises for higher n–point functions at the one-loop level and, with suitable modifications, also at higher loop orders [15]. One might have anticipated that the RA basis would be significantly easier to use compared to the Keldysh basis: there are only two RA propagators Δ_R and

Δ_A, whereas the Keldysh basis also contains the dependent combination $\Delta_S \sim \coth^\eta(x/2)[\Delta_R - \Delta_A]$, and also two RA n–point functions $\mathcal{R}_{RR\cdots R}$ and $\mathcal{R}_{AA\cdots A}$ vanish identically in general, while in the Keldysh basis only $\mathcal{K}_{11\cdots 1}$ does so. These features of the RA basis are certainly an advantage at higher loop orders. However, the vanishing of the bare Keldysh vertices with an even number of "2" indices, together with the particular form of Δ_S which arises directly in the result of Eq. (78), simplifies calculations in the Keldysh basis considerably. At least to lower orders, then, the use of the Keldysh basis and the RA basis as outlined here are comparable in difficulty.

In the $F\bar{F}$ basis the retarded self-energy could be found as $\theta(p_0)\mathcal{F}_{FF}(p,-p)$. To show the final result of this calculation for Eq. (27) has the form as Eq. (78) is somewhat tedious. An easier problem is to compute the imaginary part of this self-energy, which is discussed in detail by Weldon in the imaginary-time formalism [23]. When this is found using Eq. (27) in the $F\bar{F}$ basis, terms such as

$$\begin{aligned}
&\int dk_0 \; f(k_0,q_0)\left[\Delta_F(k)\Delta_F(q) + \Delta_F^*(k)\Delta_F^*(q)\right] = \\
&\quad = \int dk_0 \; f(q_0,r_0)\left[\Delta^+(k)\Delta^-(q) + \Delta^-(k)\Delta^+(q)\right], \\
&\int dk_0 \; g(k_0,q_0)\left[\Delta_F(k)\Delta_F^*(q) + \Delta_F^*(k)\Delta_F(q)\right] = \\
&\quad = \int dk_0 \; g(k_0,q_0)\left[\Delta^+(k)\Delta^+(q) + \Delta^-(k)\Delta^-(q)\right],
\end{aligned} \tag{79}$$

arise [17], where, for example, $\Delta^\pm(p) = 2\pi\theta(\pm p_0)\delta(p^2 - m^2)$ for bosons. Thus, θ functions of energy automatically appear which, when applied to the vertices $\widehat{g}_{fgh}(p,q,r)$, lead to simplifications due to the normalization and complex conjugation relations described previously for the $F\bar{F}$ functions. The result in the end, of course, agrees with the corresponding one of Weldon and others [15, 18, 23, 24].

For this particular calculation of the imaginary part the choice of parameters $c(p) = e^x n(x)$ and $d(p) = \eta$ in the $F\bar{F}$ basis is somewhat easier to use. However, in this example other methods such as the imaginary-time formalism or the RA or Keldysh bases are simpler, although this may not be so in more complex situations. In this regard, we remark that the steps used in this calculation are suggestive of the Cutkosky or cutting rules for the imaginary parts of graphs as derived in the approach of 't Hooft and Veltman [17]. These rules at finite temperature using the $+/-$ D_{ab} propagators have

been discussed in Refs. [18, 25, 26, 27], and it is straightforward to derive the corresponding rules for the $F\bar{F}$ basis. A feature worth noting about such a derivation is that, because the zero temperature Feynman propagator and its complex conjugate are involved, the "non-cuttable" graphs that arise beyond the one-loop level at finite temperature when the D_{ab} propagators are used will not be present in the $F\bar{F}$ rules; for this the "causality" relations of Eq. (74) play a crucial role.

7 Discussion

In many instances physical quantities are more naturally expressed in terms of Green functions other than the usual time-ordered products. At finite temperature it is important to be aware of which function is needed, as differences between the various types of products is more pronounced than at zero temperature [10, 11]. Although in principle it is possible to extract all such information from the original D_{ab} basis of the real-time formalism, in practice this may not be the most convenient method, especially in light of the constraints of Eqs. (11, 12) among the components of D_{ab}. For this reason formalisms such as the Keldysh and Retarded/Advanced bases have been constructed which involve explicitly, in particular, the retarded propagator. The transformation of Eq. (9) from the D_{ab} basis to a generic transformed $\widehat{D}_{XY}$ basis provides a unifying framework for these various real-time formalisms, in that the difference between them can be attributed to the different ways the constraints of Eqs. (11, 12) are implemented.

We have studied here three classes of such transformations – the Keldysh basis, the retarded/advanced RA basis, and a Feynman-like $F\bar{F}$ basis – which in a sense are attempts to implement the constraints of Eqs. (11, 12) "economically". By means of the transformation between the bases we have also seen how information in one basis can be translated into the equivalent information in another basis; this enabled us, for example, to examine the relation between various real-time functions and certain analytically continued imaginary-time functions. Finally, we compared some aspects of these bases as they arise in practice. Although some of the relations in the $F\bar{F}$ basis such as complex conjugation can be chosen to be simpler than the corresponding ones in the RA basis, it appears in general that the conditions of Eqs. (11, 12) are implemented more efficiently in actual calculations in

the RA basis [15, 28, 29, 30], or even, comparable at least to lower orders, in the Keldysh basis. Thus, for practical purposes, it appears that of these three bases the RA or perhaps the Keldysh basis is easier to use in general, although this could depend on the particular application and may also be a matter of taste.

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References

[1] A. L. Fetter and J. D. Walecka, *Quantum Theory of Many-Particle Systems*, (McGraw-Hill, New York, 1971).

[2] N. P. Landsman and Ch. G. van Weert, Phys. Rep. **145**, 141 (1987).

[3] J. Kapusta, *Finite Temperature Field Theory*, (Cambridge University Press, Cambridge, 1989).

[4] H. Umezawa, H. Matsumoto, and M. Tachiki, *Thermo Field Dynamics and Condensed States*, (North Holland, Amsterdam, 1982).

[5] Ch. G. van Weert, "Aspects of thermal field theory", in *Banff/CAP Workshop on Thermal Field Theory*, edited by F. Khanna, R. Kobes, G. Kunstatter, and H. Umezawa (World Scientific, Singapore, 1994).

[6] J. Schwinger, J. Math. Phys. **2**, 407 (1961).

[7] L. V. Keldysh, Sov. Phys. JETP **20**, 1018 (1964).

[8] R. A. Craig, J. Math. Phys. **9**, 605 (1968).

[9] K. Chou, Z. Su, B. Hao, and L. Yu, Phys. Rep. **118**, 1 (1985).

[10] T. S. Evans, Phys. Lett. **B252**, 108 (1990); *ibid* **B249**, 286 (1990).

[11] T. S. Evans, Nucl. Phys. **B374**, 340 (1992).

[12] R. Baier and A. Niégawa, Phys. Rev. **D49**, 4107 (1994).

[13] F. Guerin, Nice preprint INLN 1993/14.

[14] P. A. Henning and H. Umezawa, Phys. Lett. **B303**, 209 (1993); Nucl. Phys. **B417**, 463 (1994).

[15] P. Aurenche and T. Becherrawy, Nucl. Phys. **B379**, 259 (1992).

[16] M. A. van Eijck and Ch. G. van Weert, Phys. Lett. **B278**, 305 (1992).

[17] G. 't Hooft and M. Veltman, in *Particle Interactions at Very High Energies*, edited by D. Speiser, F. Halzen, and J. Weyers (Plenum, New York, 1974), Part B, p. 177.

[18] R. L. Kobes and G. W. Semenoff, Nucl. Phys. **B260**, 714 (1985); *ibid* **B272**, 329 (1986).

[19] R. Kobes, Phys. Rev. **B45**, 3230 (1992).

[20] J. C. Taylor, Phys. Rev. **D47**, 725 (1993).

[21] R. Baier, B. Pire, and D. Schiff, Phys. Lett. **B238**, 367 (1990).

[22] R. Kobes, Phys. Rev. Lett. **67**, 1384 (1991).

[23] H. A. Weldon, Phys. Rev. **D28**, 2007 (1983).

[24] F. Guerin, Phys. Rev. **D49**, 4182 (1994).

[25] R. Kobes, Phys. Rev. **D43**, 1269 (1991).

[26] N. Ashida, H. Nakkagawa, A. Niégawa, and H. Yokota, Ann. Phys. (N. Y.) **215**, 315 (1992); Phys. Rev. **D45**, 2066 (1992).

[27] S. Jeon, Phys. Rev. **D47**, 4568 (1993).

[28] P. Aurenche, E. Petitgirard, and T. del Rio Gatzelurrutia, Phys. Lett. **B297**, 337 (1992).

[29] P. Aurenche, T. Becherrawy, and E. Petitgirard, ENSLAPP preprint A–452/93.

[30] M. A. van Eijck, Can. J. Phys. **71**, 237 (1993); M. A. van Eijck, C. R. Stephens, and Ch. G. van Weert, Mod. Phys. Lett. **A9**, 309 (1994).